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HEAT TRANSFER BY LAMINAR FLOW IN A CIRCULAR PIPE UNDER TRANSVERSE MAGNETIC FIELD

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Abstract—In this paper the heat-transfer problem of a conducting liquid (forced convection) in a circular pipe under a transverse magnetic field is studied when the walls of the pipe are kept at a constant axial temperature gradient. The mean temperature and the Nusselt numbers are presented for small values of the Hartmann number and are shown graphically.

NOMENCLATURE

A, constant axial temperature gradient;
a, radius of the pipe;
H_0, uniform transverse magnetic field;
I_n, modified Bessel function of first kind and n^{th} order;
$k = M/2$, half the Hartmann number;
k_1, thermometric conductivity;
K, fluid thermal conductivity;
M, Hartmann number $= \mu H_0 a \left(\frac{\sigma}{\eta}\right)^{\frac{1}{2}}$;
Nu, Nusselt number;
P_0, dimensionless axial pressure gradient;
Pe, Peclet number;
r', radial co-ordinate;
r, dimensionless radial co-ordinate;
T, temperature of the fluid;
T_m, mean temperature of the fluid;
T_w, temperature of the wall;
v_0, the mean velocity of the fluid;
w', axial component of fluid velocity;
w, dimensionless axial component of fluid velocity;
z' axial co-ordinate;
z, dimensionless axial co-ordinate;
ρ, density of the fluid;
μ, permeability of the magnetic field;
σ, electric conductivity of the fluid;
η, coefficient of viscosity of the fluid;
ν, kinematic coefficient of viscosity of the fluid;
θ, circumferential co-ordinate.

INTRODUCTION

THE study of heat transfer for an electrically conducting fluid under the influence of a magnetic field is now considered of significant importance due to its application in many engineering problems such as the magneto-hydrodynamic generator, plasma studies, nuclear reactors and those dealing with liquid metals. These applications approximate to a linearly varying wall temperature or uniform heat flux rather than a uniform wall temperature. Nusselt [1] was the first to discuss the problem of heat transfer for the non-magnetic case in Poiseuille flow for uniform wall heat flux. Seigel [2] has solved the similar problem for a parallel plate channel corresponding to a Hartmann velocity profile for the magnetic case. Nigam and Singh [3] have studied the same problem when the plates are kept at a uniform temperature. However the problem for a circular pipe is more important from a practical point of view.

In the present paper, the problem of heat transfer for the fully developed laminar flow through a circular pipe for an electrically conducting, incompressible, viscous fluid, under the action of a uniform transverse magnetic field is presented when the walls of the pipe are kept at a constant temperature gradient. The heat generated due to viscous and electric dissipation is neglected. It is assumed that even with a moderate velocity the difference in the wall temperature and fluid temperature is small enough to permit the neglect of buoyancy force in comparison with inertia and frictional forces

It is found that the mean temperature increases and the Nusselt number increases as the intensity of the applied magnetic field is increased. Similar results have already been obtained by Nigam and Singh [3] in case of Hartmann flow for a flat plate. It is also noted that the mean temperature and the Nusselt number both decrease as θ increases from zero to $\pi/2$. The results are consistent with the conclusions drawn by Singh and Nariboli [4] that the velocity profiles for $\theta = 0$ are nearly identical with those obtained by Hartmann in case of flow through parallel plates under transverse magnetic field, and for $\theta = \pi/2$ they are of parabolic nature.

STATEMENT OF THE PROBLEM AND DIFFERENTIAL EQUATION

Consider a viscous, incompressible, electrically conducting and heat conducting fluid in a fully developed laminar flow through an infinite circular pipe of radius a, under the action of a constant pressure gradient, represented non-dimensionally by P_o, in the direction of motion and a constant transverse magnetic field H_o. The walls of the pipe are kept at a constant axial temperature gradient A. Fig. 1 shows the configuration and co-ordinate system and Fig. 2 is the cross-section of the pipe showing the magnetic lines of force.

The problem of velocity distribution in this case has already been solved by Shercliff [5], Uhlenbusch and Fisher [6], Ufyland [7], Gold [8] and Singh and Nariboli [4]. The expression for the axial velocity component [4] is given as

$$\frac{4kw}{P_o} = a_o I_o(kr)\,(e^{-kr\cos\theta} + e^{kr\cos\theta}) + 2\sum_{n=1}^{\infty} a_n I_n(kr)\cos n\theta\, e^{-kr\cos\theta} + 2\sum_{n=1}^{\infty}(-1)^n a_n I_n(kr)\cos n\theta\, e^{kr\cos\theta}, \quad (1)$$

where

$$a_n = \frac{\mathrm{d}I_n(k)}{\mathrm{d}k} \Big/ I_n(k).$$

The energy equation for the associated heat-transfer problem [9] simplifies to

$$w'\frac{\partial T}{\partial z'} = k_1\left(\frac{\partial^2 T}{\partial r'^2} + \frac{1}{r'}\frac{\partial T}{\partial r'} + \frac{1}{r'^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z'^2}\right). \quad (2)$$

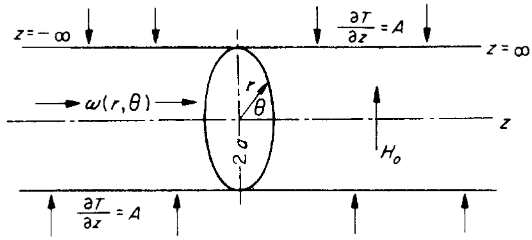

FIG. 1. Configuration and co-ordinate system.

The terms due to viscous dissipation and electric dissipation are neglected.

Introducing the dimensionless variables z, r and w defined by

$$z'/a = z, \quad w'/v_o = w$$
$$r'/a = r, \quad Pe = \frac{av_o}{k_1}, \quad (3)$$

equation (2) becomes

$$Pe\, w\frac{\partial T}{\partial z} = \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial r^2}\right). \quad (4)$$

Assuming the temperature distribution T of the form

$$T = Az + g(r, \theta) \quad (5)$$

with the condition

$$g(1, \theta) = 0; \quad (6)$$

and substituting in equation (4), the differential equation

$$Pe\, Aw = \frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} \quad (7)$$

is obtained, with the boundary condition

$$g(1, \theta) = 0. \quad (8)$$

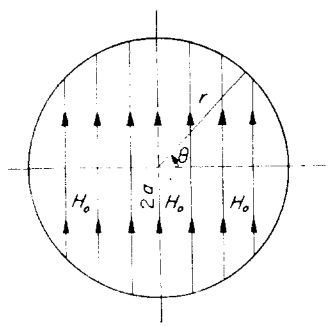

FIG. 2. Cross section showing the magnetic line of force.

The mean temperature and the Nusselt number are defined as

$$\left.\begin{aligned} T_m &= \int_0^{2\pi}\int_0^1 T\cdot r\,\mathrm{d}r\,\mathrm{d}\theta \Big/ \int_0^{2\pi}\int_0^1 r\,\mathrm{d}\theta\,\mathrm{d}r, \\ Nu &= -2K\left(\frac{\partial T}{\partial r}\right)_{r=1} \Big/ K\,(T_m - T_w). \end{aligned}\right\} \quad (9)$$

MATHEMATICAL SOLUTION

Substituting the value of w in (7), the differential equation becomes

$$\frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} = \frac{P_o\,Pe\,A}{4k} \{a_o I_o(kr)\,(e^{-kr\cos\theta} + e^{kr\cos\theta}) + 2\sum_{n=1}^{\infty} a_n I_n(kr)\cos n\theta\, [e^{-kr\cos\theta} + (-1)^n e^{kr\cos\theta}]\}. \quad (10)$$

The solution of this equation will be of the form

$$g = \sum_{n=1}^{\infty} f_m(r)\cos m\theta. \quad (11)$$

With this value of g, the equation (10) becomes

$$\sum_{m=0}^{\infty}\left[\frac{\mathrm{d}^2 f_m(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f_m(r)}{\mathrm{d}r} - \frac{m^2}{r^2} f_m(r)\right]\cos m\theta = \frac{P_o\,Pe\,A}{4k}\,[a_o I_o(kr)\,(e^{-kr\cos\theta} + e^{kr\cos\theta}) + 2\sum_{n=1}^{\infty} a_n I_n(kr)\cos n\theta\, e^{-kr\cos\theta} + 2\sum_{n=1}^{\infty} a_n(-1)^n I_n(kr)\cos n\theta\, e^{kr\cos\theta}]. \quad (12)$$

Multiplying the equation (12) by $\cos m\theta$ and integrating between the limits 0 to π,

$$\frac{\mathrm{d}^2 f_m}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f_m}{\mathrm{d}r} - \frac{m^2}{r^2} f_m = \frac{P_o\,Pe\,A}{k}\{a_o I_o(kr) I_m(kr) + \sum_{n=1}^{\infty} a_n(-1)^n [I_n(kr) I_{n+m}(kr) + I_n(kr) I_{m-n}(kr)]\} \quad (13)$$

for even m. When m is odd, the expression for the right-hand side vanishes.

The boundary condition becomes

$$f_m(r) = 0 \text{ at } r = 1,\ m = 0, 2, 4, 6, \ldots \quad (14)$$

For $m = 0$, equation (13) becomes

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}f_o}{\mathrm{d}r}\right) = \frac{P_o\,Pe\,A}{k}\{r a_o I_0^2(kr) + 2\sum_{n=1}^{\infty} a_n(-1)^n r[I_n^2(kr)]\}. \quad (15)$$

Integrating twice with respect to r and using the finiteness condition at $r = 0$ and the boundary condition (14), $f_o(r)$ is obtained as [Appendix (i)]

$$\begin{aligned} f_o(r) = \frac{P_o\,Pe\,A}{k}\Bigg(&\frac{a_o}{4}\{[I_o^2(kr) - 2I_1^2(kr) + I_o(kr) I_2(kr)]\,r^2 - [I_o^2(k) - 2I_1^2(k) + I_o(k) I_2(k)]\} \\ &- \sum_{n=1}^{\infty}(-1)^n\frac{a_n}{2}\{[I_{n+1}^2(kr) - I_{n+2}(kr) I_n(kr) - I_n^2(kr) + I_{n+1}(kr) I_{n-1}(kr)]\,r^2 \\ &- [I_{n+1}^2(k) - I_{n+2}(k) I_n(k) - I_n^2(k) + I_{n+1}(k) I_{n-1}(k)]\} \\ &- \sum_{n=1}^{\infty}\frac{2na_n(-1)^n}{k^2}\sum_{s=0}^{\infty}\frac{[(kr/2)^{2s+2n+2} - (k/2)^{2s+2n+2}]\,(2n+2s+1)!}{(2n+s+1)!\,(n+s+1)!\,(n+s)!\,s!\,(n+s+1)}\Bigg). \end{aligned} \quad (16)$$

For non-zero values of m, the solution of equation (13) is obtained by the method of variation of parameters and the two constants of integration are determined by the finiteness condition at $r = 0$ and the boundary condition (14). Thus [Appendix (ii)]

$$f_m(r) = \frac{Pe\,P_0\,A}{4km(m^2-1)} \{a_0 r^2 [(m+1) I_1(kr) I_{m-1}(kr) + (m-1) I_1(kr) I_{m+1}(kr) - 2m I_0(kr) I_m(kr)]$$
$$- a_0 r^m [(m+1) I_1(k) I_{m-1}(k) + (m-1) I_1(k) I_{m+1}(k) - 2m I_0(k) I_m(k)]$$
$$+ \sum_{n=1}^{\infty} (-1)^n a_n r^2 [(m+1) I_{m+n-1}(kr) I_{n+1}(kr) + (m+1) I_{m-n-1}(kr) I_{n-1}(kr)$$
$$+ (m-1) I_{m+n+1}(kr) I_{n-1}(kr) + (m-1) I_{m-n+1}(kr) I_{n+1}(kr) - 2m I_{m+n}(kr) I_n(kr)$$
$$- 2m I_{m-n}(kr) I_n(kr)] - \sum_{n=1}^{\infty} (-1) a_n r^m [(m+1) I_{m+n-1}(k) I_{n+1}(k)$$
$$+ (m+1) I_{m-n-1}(k) I_{n-1}(k) + (m-1) I_{m+n+1}(k) I_{n-1}(k) + (m-1) I_{m-n+1}(k) I_{n+1}(k)$$
$$- 2m I_{m+n}(k) I_n(k) - 2m I_{m-n}(k) I_n(k)]\}. \tag{17}$$

And hence,

$$T = Az + \sum_{m=0}^{\infty} f_m(r) \cos m\theta, \tag{18}$$

where m is an even integer and $f_0(r), f_2(r), f_4(r), \ldots$ etc. are given by equations (16) and (17).

In the limiting case as k tends to zero, the temperature distribution T approaches the value for the non-magnetic case given by Nusselt [1] as

$$Az - \frac{P_0\,Pe\,A}{2} \left[\frac{3}{16} - \frac{r^2}{4} + \frac{r^4}{16}\right].$$

The mean temperature and the Nusselt number have been calculated by expanding the temperature distribution T as a polynomial in kr and retaining the terms only up to sixth degrees in kr. The value of T_m is obtained [Appendix (iii)] as

$$T_m = Az + \frac{P_0\,Pe\,A}{k} \times \mathrm{D} \tag{19}$$

where

$$D = a_0 \left[\frac{1}{8} + \frac{(k/2)^2}{12} + \frac{(k/2)^4}{32} + \frac{(k/2)^6}{144}\right] + a_1 \left[\frac{(k/2)^2}{12} + \frac{(k/2)^4}{24} + \frac{(k/2)^6}{96}\right]$$
$$- a_2 \left[\frac{(k/2)^4}{96} + \frac{(k/2)^6}{240}\right] + a_3 \left[\frac{(k/2)^6}{1440}\right]$$
$$+ \frac{1}{6\pi} \left\{ - a_0 \left[\frac{(k/2)^2}{12} + \frac{(k/2)^4}{16} + \frac{3(k/2)^6}{160}\right] + a_1 \left[\frac{(k/2)^2}{6} + \frac{7(k/2)^4}{64} + \frac{13(k/2)^6}{400}\right]\right.$$
$$\left. - a_2 \left[\frac{(k/2)^2}{12} + \frac{(k/2)^4}{16} + \frac{(k/2)^6}{50}\right] + a_3 \left[\frac{(k/2)^4}{64} + \frac{3(k/2)^6}{400}\right] - a_4 \left[\frac{(k/2)^6}{800}\right]\right\}$$
$$+ \frac{1}{60 \times \pi} \left\{ - a_0 \left[\frac{(k/2)^4}{48} + \frac{(k/2)^6}{60}\right] + a_1 \left[\frac{(k/2)^4}{12} + \frac{2(k/2)^6}{45}\right] - a_2 \left[\frac{(k/2)^4}{8} + \frac{(k/2)^6}{18}\right]\right.$$
$$\left. + a_3 \left[\frac{(k/2)^4}{12} + \frac{(k/2)^6}{24}\right] - a^4 \left[\frac{(k/2)^4}{48} + \frac{(k/2)^6}{60}\right]\right\}$$
$$+ \frac{1}{240 \times \pi} \left[- a_0 \frac{(k/2)^6}{960} + a_1 \frac{(k/2)^6}{160} - a_2 \frac{(k/2)^6}{64} + a_3 \frac{(k/2)^6}{48} - a^4 \frac{(k/2)^6}{64}\right]. \tag{20}$$

The Nusselt number is given by [Appendix (iv)] $Nu = -N/D$, where

$$
\begin{aligned}
N = {} & a_o \left[1 + (k/2)^2 + (k/2)^4 + \frac{5}{18}(k/2)^6\right] - a_1 \left[(k/2)^2 + \frac{2}{3}(k/2)^4 + \frac{5}{24}(k/2)^6\right] \\
& + a_2 \left[\frac{(k/2)^4}{6} + \frac{(k/2)^6}{12}\right] - a_3 \left[\frac{(k/2)^6}{72}\right] \\
& + \frac{\cos 2\theta}{12}\left\{a_o \left[2(k/2)^2 + 2(k/2)^4 + \frac{3}{4}(k/2)^6\right] - a_1 \left[4(k/2)^2 + \frac{7}{2}(k/2)^4 + \frac{13}{10}(k/2)^6\right]\right. \\
& \left. + a_2 \left[2(k/2)^2 + 2(k/2)^4 + \frac{4}{5}(k/2)^6\right] - a_3 \left[\frac{(k/2)^4}{2} + \frac{3}{10}(k/2)^6\right] + a_4 \left[\frac{(k/2)^6}{20}\right]\right\} \\
& + \frac{\cos 4\theta}{120}\left\{a_o [(k/2)^4 + (k/2)^6] - a_1 \left[4(k/2)^4 + \frac{8}{3}(k/2)^6\right] + a_2 \left[6(k/2)^4 + \frac{10}{3}(k/2)^6\right]\right. \\
& \left. - a_3 \left[4(k/2)^4 + \frac{5}{2}(k/2)^6\right] + a^4 [(k/2)^4 + (k/2)^6]\right\} \\
& + \frac{\cos 6\theta}{420}\left[a_o \frac{(k/2)^6}{12} - a_1 \frac{(k/2)^6}{2} + \frac{5}{4}a_2 (k/2)^6 - \frac{5}{3}a_3 (k/2)^6 + \frac{5}{4}a^4 (k/2)^6\right]
\end{aligned}
\tag{21}
$$

and D is given by equation (20).

The Nusselt number also tends to 6 as $k \to 0$ in agreement with the result given by Nusselt [1].

CONCLUSIONS

The mean mixed temperature and the local Nusselt numbers are calculated for small values of the Hartmann number $M = 0{\cdot}8$, 2, 2·8 and 4, correct to the third decimal place. These results are based on the expansion of the modified Bessel functions in ascending powers of kr up to sixth degree and are valid only for small values of k. For large values of k, the asymptotic expansion of the modified Bessel functions should be used. As the Hartmann number varies from 0·8 to 4, the mean temperature increases from $-0{\cdot}041 + Az$ to $-0{\cdot}031 + Az$. For $M = 0{\cdot}8$ there is no appreciable variation in the Nusselt numbers for different values of θ, but for other values of M, they go on decreasing with the increase of θ. But for the same value of θ, the Nusselt numbers increase with M. Fig. 3 is the plot of the Nusselt number against θ for different values of M. At $\theta = 0$ the Nusselt number for different values of M, matches with the values calculated from Seigel's [2] result. For example, at $M = 2$, the Nusselt number in case of flat plate [2] is 6·92 whereas in this case the corresponding Nusselt numbers at $\theta = 0$ and $\theta = \pi/2$ are 6·7 and 6·5 respectively. Hence this shows that the temperature profiles at $\theta = 0$ are similar to those for a flat plate and those at $\theta = \pi/2$ are of parabolic nature.

The problem, when the walls of the tube are kept at constant temperature [3] will be presented in the next communication.

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REFERENCES

1. S. Goldstein, *Modern Developments in Fluid Dynamics*, Vol. II, p. 622. Clarendon Press, Oxford (1952).
2. R. Seigel, Effect of magnetic field on forced convection heat transfer in a parallel plate channel, *J. Appl. Mech.* **25**, 415 (1958).
3. S. D. Nigam and S. N. Singh, Heat transfer by laminar flow between parallel plates under the action of transverse magnetic field, *Quart. J. Mech. Appl. Math.* **13**, Part I, 850 (1960).
4. S. N. Singh and G. A. Nariboli, The flow of conducting fluids in circular pipes under transverse magnetic fields, *Appl. Sci. Res.* (in press).
5. J. A. Shercliff, The flow of conducting fluids in circular pipes under transverse magnetic fields, *J. Fluid Mech.* **1**, Part 6, 644 (1956).

6. J. Uhlenbusch and E. Fisher, *Z. Phys.* **164**, 190 (1961).
7. U. S. Ufylyand, Hartmann problem for a circular tube, Translated from *Z. Tekh. Fiz.* **30**, No. 10, 1258 (1960); Translation: *Sov. Phys. Tech. Phys.* **5**, 1194 (1961).
8. R. R. Gold, Magneto hydrodynamic pipe flow, *J. Fluid Mech.* **13**, Part 4, 505 (1962).
9. S. I. Pai, Energy equation of magneto-gas dynamics, *Phys. Rev.* **105**, No. 5, 1524 (1957).
10. N. W. McLachlan, *Bessel Functions for Engineers.* Clarendon Press, Oxford (1955).
11. G. N. Watson, *A Treatise on the Theory of Bessel Functions.* University Press, Cambridge (1958).

APPENDIX

(i) Integration of equation (15) with respect to r [[10], equations (164) and (193), pp. 201 and 203] is

$$\left.\begin{aligned}\frac{\mathrm{d}f_o}{\mathrm{d}r} = \frac{P_o Pe A}{k}\Big\{&\frac{a_o}{2} r\,[I_0^2(kr) - I_1^2(kr)] \\ &+ \sum_{n=1}^{\infty} a_n (-1)^n \Big[r I_n^2(kr) - r I_{n+1}^2(kr) \\ &- \frac{2n}{k} I_n(kr)\, I_{n+1}^2(kr)\Big] + \frac{B_1}{r}\Big\},\end{aligned}\right\} \quad (22)$$

where B_1 is the constant of integration.

Substituting the value of $I_n(kr)\, I_{n+1}(kr)$ from Watson [11, equation (5.40, 5), p. 147], the equation becomes

$$\left.\begin{aligned}\frac{\mathrm{d}f_o(r)}{\mathrm{d}r} = \frac{P_o Pe A}{k}\Big\{&r\frac{a_o}{2}\,[I_o^2(kr) - I_1^2(kr)] \\ &+ \sum_{n=1}^{\infty} r a_n (-1)^n\, [I_n^2(kr) - I_{n+1}^2(kr)] \\ &- \sum_{n=1}^{\infty} \frac{2a_n(-1)^n n}{k} \\ &\sum_{s=0}^{\infty} \frac{(kr/2)^{2s+2n+1}(2s+2n+1)!}{(2n+s+1)!\,(n+s+1)!\,(n+s)!\,s!} \\ &+ \frac{B_1}{r}\Big\}.\end{aligned}\right\} \quad (23)$$

Again integrating (23) with respect to r, $f_o(r)$ is obtained. B_1 is zero due to finiteness condition at $r = 0$, and the boundary condition (14) gives the value of the other constant C. Substitution of this value of constant in the integrated equation, gives (16).

(ii) When m is not zero, equation (13) can be written as

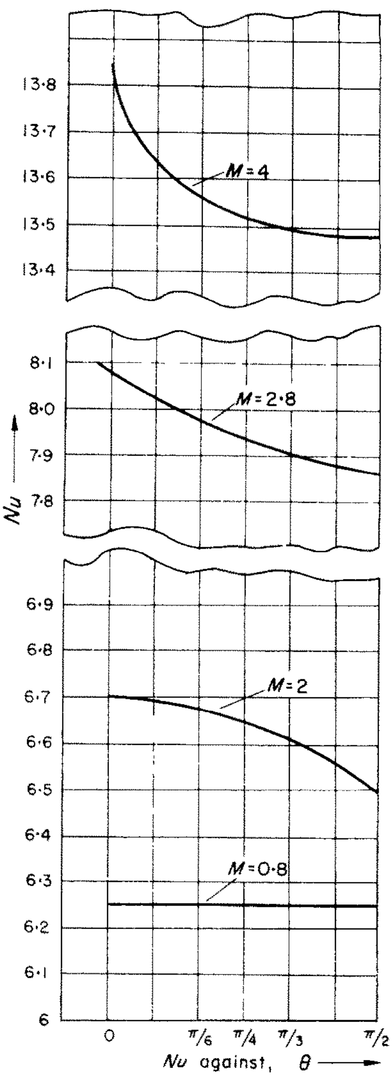

FIG. 3.

$$r^2 \frac{\mathrm{d}^2 f_m}{\mathrm{d}r^2} + r\frac{\mathrm{d}f_m}{\mathrm{d}r} - m^2 f_m = r^2 R \quad (24)$$

where R stands for the right-hand side of (13).

By the method of variation of parameters, the solution is obtained in the form

$$f_m(r) = P_1 r^m + Q_1 r^{-m}$$

where

$$\left.\begin{aligned} P_1 &= \int \frac{r^{-(m+1)} R\,\mathrm{d}r}{2m} + B_2 \\ Q_1 &= -\int \frac{r^{(m-1)} R\,\mathrm{d}r}{2m} + C_2 \end{aligned}\right\} \quad (25)$$

and B_2 and C_2 are constants of integration.

To evaluate P_1 and Q_1 the following integrals are derived.

The results

$$\frac{d}{dx}[x^\lambda I_u(x) I_v(x)] = x^\lambda [I_u(x) I_{v+1}(x) + I_{u+1}(x) I_v(x)] + (\lambda + u + v) x^{\lambda-1} I_u(x) I_v(x). \quad (26)$$

and

$$\frac{d}{dx}[x^\lambda I_{u+1}(x) I_{v+1}(x)] = x^\lambda [I_u(x) I_{v+1}(x) + I_{u+1}(x) I_v(x)] + (\lambda - u - v - 2) x^{\lambda-1} I_{u+1}(x) I_{v+1}(x) \quad (27)$$

which are obtained by equations (164) and (165), p. 20, McLachlan [10], when subtracted, give

$$\frac{d}{dx}\{x^\lambda [I_u(x) I_v(x) - I_{u+1}(x) I_{v+1}(x)]\} = (\lambda + u + v) x^{\lambda-1} I_u(x) I_v(x) - (\lambda - u - v - 2) x^{\lambda-1} I_{u+1}(x) I_{v+1}(x). \quad (28)$$

Integrating and giving special values to λ, i.e., $\lambda = -(u + v)$ and $\lambda = (u + v + 2)$, the integrals are obtained,

$$\left.\begin{aligned} &\int x^{u+v+1} I_u(x) I_v(x)\,dx = \frac{I_u(x) I_v(x) - I_{u+1}(x) I_{v+1}(x)}{2(u+v+1)} x^{u+v+2} \\ &\int x^{-u-v+1} I_u(x) I_v(x)\,dx = \frac{I_{u-1}(x) I_{v-1}(x) - I_u(x) I_v(x)}{2(u+v-1)} x^{-u-v+2} \end{aligned}\right\} \quad (29)$$

If R is written as

$$\frac{P_o Pe A}{k}\{a_o I_o(kr) I_m(kr) + \sum_{n=1}^{\infty} a_n (-1)^n [I_n(kr) I_{m+n}(kr) + I_n(kr) I_{m-n}(kr)]\}$$

in equation (25), the value of P_1 and Q_1 can be determined directly using the two integrals (29) for $u = 0, -n, n$ and $v = m, m+n$ and $m - n$.

The constants are determined by the finiteness condition at $r = 0$ which gives $C_2 = 0$ and the condition (14) which determines B_2. And thus $f_m(r)$ is obtained in the form of equation (17).

$$\text{(iii)}\quad T_m = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 Tr\,dr\,d\theta. \quad (30)$$

$$= \frac{1}{\pi}\int_0^{2\pi}\int_0^1 \sum_{m=0}^{\infty} r f_m(r) \cos m\theta\,dr\,d\theta + Az \quad (31)$$

$$= \frac{4}{\pi}\int_0^1 \sum_{m=0}^{\infty} r f_m(r)\,dr + Az, \quad (32)$$

where $f_m(r)$ is known as the product of two modified Bessel functions in the form $I_p(kr) \times I(kr)$ which can be written in ascending powers of kr by the expansion

$$I_p(kr) I_q(kr) = \sum_{s=0}^{\infty} \frac{(kr/2)^{p+q+2s}(p+q+2s)!}{(p+q+s)!(p+s)!(q+s)!s!}. \quad (33)$$

Giving special values to p and q from equation (17), and retaining the terms up to sixth degree in kr in the expanded series, the value of $f_m(r)$ is known as a polynomial in kr. Substituting this value of $f_m(r)$ in (32), and then integrating, the value of T_m is determined as given by (19) and (20).

$$\text{(iv)}\quad Nu = -2K\left(\frac{\partial T}{\partial r}\right)_{r=1} \Big/ K(T_m - T_w) \quad (34)$$

$$= -2\left(\frac{\partial T}{\partial r}\right)_{r=1} \Big/ \frac{P_o Pe A}{k} \times D, \quad (35)$$

as $T_w = Az$, and D is given by equation (20).

But,

$$\left(\frac{\partial T}{\partial r}\right)_{r=1} = \sum_{m=0}^{\infty}\left(\frac{df_m(r)}{dr}\right)_{r=1} \cos m\theta, \quad (36)$$

where $f_m(r)$ can be expanded in ascending powers of kr as in (iii) above and can be written as a polynomial of sixth degree in kr. Differentiating term by term, the value of $df_m(r)/dr$ can be calculated at $r = 1$. Substituting this value in (35) the value of Nusselt number is determined as in equation (21).

Zusammenfassung—Der Wärmeübergang in einer elektrisch leitenden Flüssigkeit (Zwangskonvektion) in einem Kreisrohr mit radial angelegtem Magnetfeld wird untersucht. Dabei weisen die Rohrwände einen konstanten, achsialen Temperaturgradienten auf. Die mittlere Temperatur und die Nusselt-Zahlen sind für kleine Werte der Hartmann-Zahl angegeben und grafisch dargestellt.

Аннотация—В настоящей статье рассматривается проблема теплообмена теплопроводной жидкости (вынужденная конвекция) в кольцевой трубе при наличии поперечного магнитного поля и постоянного аксиального градиента температуры стенки трубы. Для малых значений числа Хартманна приведены средняя температура и числа Нуссельта, а также дано их графическое представление.